STUDENT NUMBER

GOSFORD HIGH SCHOOL

2015
TRIAL HSC EXAMINATION

EXTENSION 1 MATHEMATICS

General Instructions:

- Reading time: 5minutes
- Working time: 2 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided
- In Questions 11-14, show relevant mathematical reasoning and/or calculations

Total marks: - 70

Section I (10 marks)
Attempt Questions 1- 10.
Answer on the Multiple Choice answer sheet provided
Allow about 15 minutes for this section

Section II (60 marks)
Attempt Questions 11-14
Start each question in a separate answer booklet
Allow about 1 hour and 45 minutes for this section

MULTIPLE CHOICE	/10
QUESTION 11	/15
QUESTION 12	/15
QUESTION 13	/15
QUESTION 14	/15
TOTAL	/70

Section I.

Total marks (10).

Attempt Questions 1-10.

Allow about 15 minutes for this section.

Answer on the multiple choice answer sheet provided. Select the alternative A, B, C, D that best answers the question. Fill in the response oval completely.

1. The point A has coordinates $(-1,4)$ and the point B has coordinates $(5,-2)$. Find the coordinates of the point which divides AB externally in the ratio 1:3.

   A. $(-4,7)$ B. $(4,-7)$ C. $(7,-4)$ D. $(-7,4)$

2. The solution to the inequation $\frac{x^2-2}{x} \leq 1$ is

   A. $x \leq -1,\ x \geq 2$

   B. $-1 \leq x < 0,\ x \geq 2$

   C. $x \leq -1,\ 0 < x \leq 2$

   D. $x \leq -1,\ 0 \leq x \leq 2$

3. A committee of three is to be chosen from a group of five men and seven women. How many different committees can be formed if the committee is to contain at least one man and at least one woman?

   A. 220 B. 175 C. 70 D. 105

4. If the acute angle between the lines $2x - y = 2$ and $kx - y = 5$ is 45^0, then the value of k is

   A. 3 or $\frac{1}{3}$ B. -3 or $\frac{-1}{3}$ C. 3 or $\frac{-1}{3}$ D. -3 or $\frac{1}{3}$

5. The acceleration of a particle moving along a straight line is given by $a = -2e^{-x}$ where x metres is the displacement from the origin. If the velocity of the particle is given by v, then

   A. $v^2 = 2e^{-x} + c$ B. $v^2 = 2e^{x} + c$ C. $v^2 = 4e^{-x} + c$ D. $v^2 = 4e^{x} + c$

6. $\int \frac{1}{1+9x^2}\,dx =$

A. $\frac{1}{27}\tan^{-1}3x + c$

B. $\frac{1}{9}\tan^{-1}3x + c$

C. $\frac{1}{3}\tan^{-1}3x + c$

D. $\tan^{-1}3x + c$

7. If $\frac{dN}{dt} = -0.4(N - 100)$ and $N = 0$ when $t = 0$, the value of N correct to 2 decimal places when $t = 20$ is

A. 7.69

B. 99.97

C. 100.03

D. 192.31

8. Victoria made an error proving that $3^{2n} - 1$ is divisible by 8 (where n is an integer greater than 0) using mathematical induction. Part of the proof is shown below.

Step 2: Assume the result true for $n = k$

$3^{2k} - 1 = 8P$ *where* P *is an integer.* Line 1

Hence $3^{2k} = 8P + 1$

To prove the result is true for $n = k + 1$

$3^{2(k+1)} - 1 = 8Q$ *where* Q *is an integer.* Line 2

$$
\begin{aligned}
\text{LHS} &= 3^{2(k+1)} - 1 \\
&= 3^{2k} \times 3^2 - 1 \\
&= (8P + 1) \times 3^2 - 1 && \text{Line 3} \\
&= 72P + 1 - 1 && \text{Line 4} \\
&= 72P \\
&= 8(9P) \\
&= 8Q \\
&= \text{RHS}
\end{aligned}
$$

Which line did Victoria make an error?

(A) Line 1

(B) Line 2

(C) Line 3

(D) Line 4

9. Which of the following is an expression for $\int 2cos^2 x\, dx$.

A. $x - \frac{1}{2} sin2x + c$

B. $x + \frac{1}{2} sin2x + c$

C. $x - sin2x + c$

D. $x + sin2x + c$

10. One approximation to the solution of the equation $\frac{\pi}{4} + \tan^{-1} x - x^2 = 0$ is $x = 1$. What is another approximation to this solution using one application of Newton's method?

A. $x = 1.3805$

B. $x = 1.3914$

C. $x = 1.4125$

D. $x = 1.4156$

Section II.

Total marks (60).

Attempt Questions 11-14.

Allow about 1 hour and 45 minutes for this section.

Answer all questions, starting each question in a separate writing booklet.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) (i) Find $\frac{d}{dx}(x \sin 2x)$. (1)

(ii) Hence or otherwise find $\int x \cos 2x \, dx$. (3)

(b) Consider the function $f(x) = 2\cos^{-1}\frac{x}{3}$.

(i) Evaluate $f(0)$. (1)

(ii) Draw the graph of $y = f(x)$. (1)

(iii) State the domain and range of $y = f(x)$. (2)

(c) If α, β and γ are the roots of $2x^3 - 6x^2 + x + 2 = 0$, find the value of

(i) $\alpha + \beta + \gamma$. (1)

(ii) $(\alpha - 1)(\beta - 1)(\gamma - 1)$. (2)

(d) Evaluate $\int_1^6 x\sqrt{x+3}\, dx$ by means of the substitution $u^2 = x + 3$. (4)

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) Two points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ lie on the parabola $x^2 = 4ay$.

(i) Show that the equation of the tangent to the parabola at P is

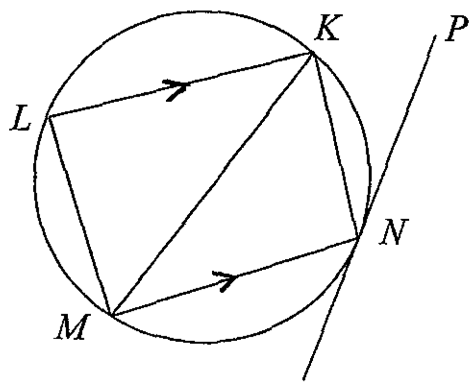
$$y = px - ap^2.$$ (2)

(ii) The tangent at P and the line through Q parallel to the y axis intersect at T. Find the coordinates of T. (2)

(iii) Write down the coordinates of M, the midpoint of PT. (1)

(iv) Determine the locus of M when $pq = -1$. (1)

(b) The diagram below shows a cyclic quadrilateral $MNKL$ with $MN \parallel LK$.



NOT TO SCALE

PN is a tangent to the circle and $\angle MNK = 2\angle KNP$.

Copy the diagram into your writing booklet and prove that ΔLMK is isosceles.

Hence, show that MK bisects $\angle LMN$. (4)

(c) The point $P(2, -1)$ divides the interval joining $A(-2,3)$ and $B(8,-7)$ internally the ratio $m:n$. Find the values of m and n. (3)

(d) Differentiate $\dfrac{cos^{-1}x}{x}$ (2)

Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) (i) Show that $\frac{dv}{dt} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$. (2)

(ii) A particle is moving along a straight line. At time, t seconds, its displacement, x metres, from a fixed point O on the line is such that $t = x^2 - 3x + 2$. Find an expression for its velocity v in terms of x. (1)

(iii) Hence, find an expression for the particle's acceleration a in terms of x. (2)

(b) (i) Express $\sqrt{3}cosx - sinx$ in the form $Rcos(x + \alpha)$ where $0 \leq \alpha \leq \frac{\pi}{2}$. (2)

(ii) Hence, or otherwise, solve $\sqrt{3}cosx - sinx = 1$. (2)

(c) How many 4-letter "words" consisting of at least one vowel and at least one consonant can be made from the letters of the word EQUATION? (2)

(d) The region bounded by the curve $y = cos\ 2x$ and the x-axis between $x = 0$ and $x = \frac{\pi}{4}$ is rotated about the x-axis. Find the exact value of the volume of the solid of revolution generated. (4)

Question 14 (15 marks) Use a SEPARATE writing booklet.

(a) Use mathematical induction to prove that for all positive integers n:

$$\sum_{r=1}^{n} r(r!) = (n+1)! - 1.$$

(4)

(b) A particle moves in a straight line so that its displacement, x metres, at time t seconds, is given by $x = 4 - 2sin^2 t$.

(i) Show that the motion is simple harmonic. (2)

(ii) Find the period and the centre of the motion. (2)

(iii) Show that the velocity v of the particle in terms of its displacement can be expressed as $v^2 = 4(-8 + 6x - x^2)$. (2)

(c) (i) Show that the range of flight of a projectile fired at an angle of α to the horizontal with velocity v is $\dfrac{v^2 sin2\alpha}{g}$ where g is the acceleration due to gravity. (2)

The equations describing the trajectory of the projectile are:

$$x = vt\cos\alpha\,,\, y = vt\sin\alpha - \frac{1}{2}gt^2.$$

(You are **NOT** required to prove these equations)

(ii) A cannon fires a shell at an angle of 45^0 to the horizontal and strikes a point $50m$ beyond its target. When fired with the same velocity at an angle of 30^0 it strikes a point $20m$ in front of the target. Calculate the horizontal distance between the cannon and the target correct to 2 decimal places. (3)

END OF PAPER

2015 TRIAL HSC: EXT 1 SOLUTIONS

1. $\dfrac{lx_1+kx_2}{k+l}$ $\quad$ $\dfrac{ly_1+ky_2}{k+l}$ $\quad$ $k:l = -1:3$

$= \dfrac{3\times -1 + -1\times 5}{-1+3}$ $\quad$ $\dfrac{3\times 4 + -1\times -2}{-1+3}$

$= \dfrac{-8}{2}$ $\quad$ $= \dfrac{+14}{2}$

$= -4$ $\quad$ $= 7$

$\therefore (-4, 7)$.

Hence (A).

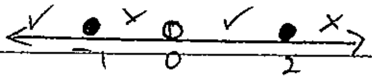
2. If $\dfrac{x^2-2}{x} = 1$, $x \neq 0$

$x^2 - 2 = x$

$x^2 - x - 2 = 0$

$(x-2)(x+1) = 0$

$\therefore x = 2,\ x = -1$


Testing pts.

$\therefore x \le -1$ or $0 < x \le 2$.

Hence (C).

3. 1M, 2W or 2M 1W

$= {}^5C_1 \times {}^7C_2 + {}^5C_2 \times {}^7C_1$

$= 105 + 70$

$= 175$

Hence (B)

4. $m_1 = 2,\ m_2 = k$

$\therefore \left|\dfrac{2-k}{1+2k}\right| = 1$

$\dfrac{2-k}{1+2k} = 1$ or $\dfrac{2-k}{1+2k} = -1$

$2-k = 1+2k \qquad 2-k = -1-2k$

$1 = 3k \qquad k = -3$

$k = \frac{1}{3}$

Hence D

5. $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -2e^{-x}$

$\frac{1}{2}v^2 = 2e^{-x} + k$

$v^2 = 4e^{-x} + C$

Hence C

6. $\int \frac{1}{1+9x^2}\,dx = \frac{1}{9}\int \frac{dx}{\frac{1}{9}+x^2}$

$= \frac{1}{9} \times \frac{1}{\frac{1}{3}} \tan^{-1} \frac{x}{\frac{1}{3}} + c$

$= \frac{1}{3}\tan^{-1} 3x + c$

Hence C

7. $\frac{dN}{dt} = -0.4(N-100)$

$N = 100 + Ae^{kt}$

$0 = 100 + Ae^{0}$

$\therefore A = -100$

$N = 100 - 100e^{-0.4t}$

$= 100 - 100e^{-0.4\times 20}$

$= 100 - 100e^{-8}$

$= 99.9664$

$\doteq 99.97$

Hence B

8. Line 4 should read

$72P + 9 - 1$

Hence D

9. Since $\cos 2x = 2\cos^2 x - 1$

$\cos 2x + 1 = 2\cos^2 x$

$\therefore I = \int 1 + \cos 2x \, dx$

$= x + \frac{1}{2}\sin 2x + c$

Hence B

10. Let $f(x) = \frac{\pi}{4} + \tan^{-1}x - x^2$

$f'(x) = \frac{1}{1+x^2} - 2x$

$f(1) = \frac{\pi}{4} + \tan^{-1}1 - 1^2$

$= \frac{\pi}{4} + \frac{\pi}{4} - 1$

$= \frac{\pi}{2} - 1$

$f'(1) = \frac{1}{1+1} - 2$

$= -1.5$

$a_2 = 1 - \frac{\frac{\pi}{2}-1}{-1.5}$

$= 1.3805\ldots$

Hence A

a) (i) Let $y = x \cdot \sin 2x$

$y' = \sin 2x \cdot 1 + x \cdot 2\cos 2x$

$\therefore \frac{d}{dx}\left[x \sin 2x\right] = \sin 2x + 2x \cos 2x$ (1)

(ii) Since $\frac{d}{dx}\left[x \sin 2x\right] = \sin 2x + 2x \cos 2x$

$x \sin 2x = \int \sin 2x \, dx + \int 2x \cos 2x \, dx$

$\therefore x \sin 2x - \int \sin 2x \, dx = 2\int x \cos 2x \, dx$

$x \sin 2x + \frac{1}{2}\cos 2x + c_1 = 2\int x \cos 2x \, dx$ (3)

$\frac{x}{2} \sin 2x + \frac{1}{4}\cos 2x + \frac{c_1}{2} = \int x \cos 2x \, dx$

$\therefore \int x \cos 2x \, dx = \frac{x}{2}\sin 2x + \frac{1}{4}\cos 2x + k$

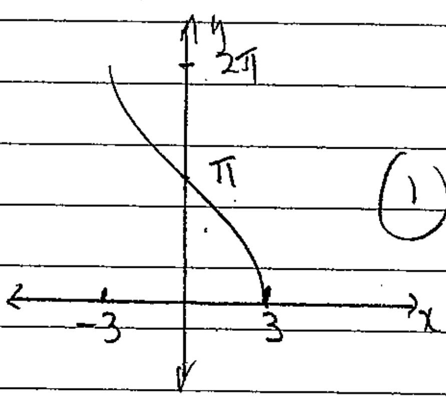
b) (i) $f(x) = 2\cos^{-1}\frac{x}{3}$

$f(0) = 2\cos^{-1} 0$ (1)

$= 2 \times \frac{\pi}{2}$

$= \pi$

(ii)



(1)

(iii) $D: -1 \le \frac{x}{3} \le 1$

ie $-3 \le x \le 3$ (1) ✓

$R: 0 \le \frac{y}{2} \le \pi$

ie $0 \le y \le 2\pi$ (1) ✓

c) $P(x) = 2x^3 - 6x^2 + x + 2$.

(i) $\alpha + \beta + \gamma = \frac{-b}{a}$

$= \frac{--6}{2}$

$= 3$ (1)

(ii) $(\alpha-1)(\beta-1)(\gamma-1) = (\alpha-1)[\beta\gamma - \beta - \gamma + 1]$

$= \alpha\beta\gamma - \alpha\beta - \alpha\gamma + \alpha - \beta\gamma + \beta + \gamma - 1$

$= \alpha\beta\gamma - (\alpha\beta + \alpha\gamma + \beta\gamma) + (\alpha+\beta+\gamma) - 1$

$= -\frac{d}{a} - \frac{c}{a} + -\frac{b}{a} - 1$

$= -\frac{2}{2} - \frac{1}{2} + 3 - 1$ (3)

$= \frac{1}{2}$.

d) Let $u^2 = x + 3 \Rightarrow x = u^2 - 3$

$2u\,du = dx$

if $x = 1$, $u = 2$

$x = 6$, $u = 3$

$\therefore I = \int_2^3 (u^2-3) \cdot u \cdot 2u\,du$

$= \int_2^3 2u^4 - 6u^2\,du$

$= \left[\frac{2u^5}{5} - \frac{6u^3}{3}\right]_2^3$ (4)

$= \left(\frac{2 \times 3^5}{5} - 54\right) - \left(\frac{2 \times 2^5}{5} - 16\right)$

$= 46\frac{2}{5}$.

2. a) (i) If $x^2 = 4ay$

$$y = \frac{x^2}{4a}$$

$$y' = \frac{2x}{4a}$$

When $x = 2ap$, $y' = \frac{4ap}{4a}$

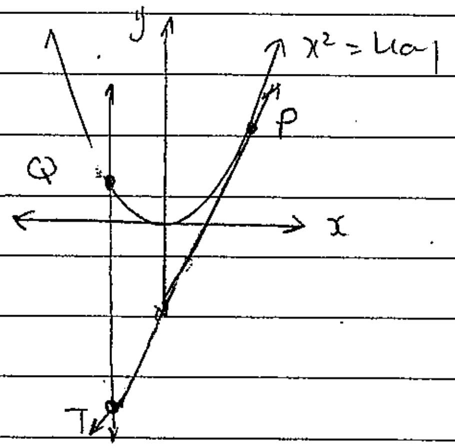
$$= p$$

Eq^n is $y - ap^2 = p(x - 2ap)$

$y - ap^2 = px - 2ap^2$ (2)

$y = px - ap^2$

(ii)



$y = px - ap^2$ — (1)

$x = 2aq$ — (2)

$\therefore$ $y = p \cdot 2aq - ap^2$ (2)

$= 2apq - ap^2$

$\therefore$ T is $(2aq, 2apq - ap^2)$

(iii) $x = \frac{2ap + 2aq}{2}$, $y = \frac{ap^2 + 2apq - ap^2}{2}$

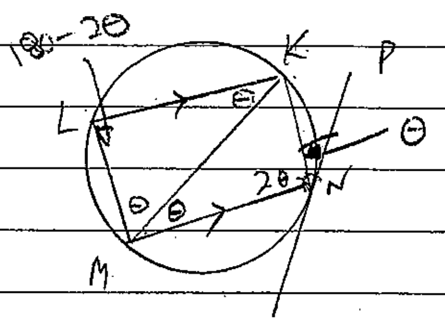
$\therefore$ M is $(a(p+q), apq)$ (1)

(iv) Since $y = apq$

$y = a \times -1$ (1)

$y = -a$.

b)



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Let $\angle KNP$ be θ

Then $\angle MNK$ is 2θ

Now $\angle NMK = \theta$ ($\angle$ in the alternate segment theorem)

$\therefore \angle LKM = \theta$ (alternate $\angle$'s in parallel lines are equal $MN \parallel LK$)

Also, $\angle MLK = 180 - 2\theta$ (opp $\angle$'s of a cyclic quad. are supplementary)

So $\angle LMK = 180 - (180 - 2\theta) - \theta$ ($\angle$ sum of a Δ is $180°$)

$= \theta$

$\therefore \Delta LMK$ is isosceles since $\angle LMK = \angle LKM = \theta$.

Also $\angle LMK = \angle NMK = \theta$

$\therefore$ MK bisects $\angle LMN$.

c) Let the ratio be $k:1$

$$\therefore \quad 2 = \frac{1 \times -2 + k \times 8}{k+1}$$

$$2 = \frac{8k - 2}{k+1}$$

$$2k + 2 = 8k - 2$$

$$4 = 6k$$

$$k = 4/6$$

$$k = 2/3$$

$\therefore$ The ratio is $\frac{2}{3} : 1$

$= 2 : 3$

3

Hence, $m = 2$, $n = 3$.

d) Let $y = \frac{\cos^{-1}x}{x}$

$$y' = \frac{vu' - uv'}{v^2}$$

$$= \frac{x \times \frac{-1}{\sqrt{1-x^2}} - \cos^{-1}x \times 1}{x^2}$$

$$= \frac{\frac{-x}{\sqrt{1-x^2}} - \cos^{-1}x}{x^2}$$

2

$$= \frac{-x - \sqrt{1-x^2} \cdot \cos^{-1}x}{x^2\sqrt{1-x^2}}.$$

13. a) (i) $\frac{dv}{dt} = \frac{dv}{dx} \times \frac{dx}{dt}$

$$= \frac{dv}{dx} \times v$$

$$= \frac{dv}{dx} \times \frac{d}{dv}\left(\tfrac{1}{2}v^2\right)$$

2

$$= \frac{d}{dx}\left(\tfrac{1}{2}v^2\right)$$

(ii) $t = x^2 - 3x + 2$

$$\frac{dt}{dx} = 2x - 3$$

$$\therefore \frac{dx}{dt} = \frac{1}{2x-3}$$

1

ie. $v = \frac{1}{2x-3}$

(iii) $a = \frac{d}{dx}\left(\tfrac{1}{2}v^2\right)$

$$= \frac{d}{dx}\left(\frac{1}{2} \times \frac{1}{(2x-3)^2}\right)$$

$= \frac{d}{dx}\left[\frac{1}{2}(2x-3)^{-2}\right]$

$= \frac{1}{2} \times -2(2x-3)^{-3} \times 2$

$= \frac{-2}{(2x-3)^3}$ 2

b) (i) $\sqrt{3}\cos x - 1\sin x$

$= \cos x . \sqrt{3} - \sin x . 1$

$= 2\cos(x+\alpha)$ where $\tan\alpha = \frac{1}{\sqrt{3}}$

$= 2\cos\left(x+\frac{\pi}{6}\right)$ 2

(ii) If $\sqrt{3}\cos x - \sin x = 1$

$2\cos\left(x+\frac{\pi}{6}\right) = 1$

$\cos\left(x+\frac{\pi}{6}\right) = \frac{1}{2}$

$x+\frac{\pi}{6} = 2n\pi \pm \frac{\pi}{3}$ 2

$\therefore x = -\frac{\pi}{6} + 2n\pi \pm \frac{\pi}{3}$

c) (i) The "words" could contain

5 vowels, 3 consonants

a) 1 v, 3 c

b) 2 v, 2 c

c) 3 v, 1 c

No of selections $= {}^5C_1 \times {}^3C_3 + {}^5C_2 \times {}^3C_2 + {}^5C_3 \times {}^3C_1$

$= 5 + 30 + 30$

$= 65$

No of words $= 65 \times 4!$

$= 1560.$ 2

d) $V = \pi \int_0^{\pi/4} \cos^2 2x \, dx$

$= \pi \int_0^{\pi/4} \frac{1}{2}\cos 4x + \frac{1}{2} \, dx$

$= \frac{\pi}{2} \int_0^{\pi/4} \cos 4x + 1 \, dx$

$= \frac{\pi}{2} \left[\frac{\sin 4x}{4} + x \right]_0^{\pi/4}$

$= \frac{\pi}{2} \left[\left(\frac{\sin \pi}{4} + \frac{\pi}{4} \right) - \left(\frac{\sin 0}{4} + 0 \right) \right]$

$= \frac{\pi}{2} \left[0 + \frac{\pi}{4} - (0 + 0) \right]$ 4

$= \frac{\pi^2}{8}$ units^3.

14. a) If $n = 1$, LHS $= 1(1!)$
$= 1$
RHS $= 2! - 1$
$= 1$

$\therefore$ True for $n = 1$

Assume true for $n = k$

ie $1(1!) + 2(2!) + \dots k(k!) = (k+1)! - 1$, k is a positive integer

Prove true for $n = k+1$.

ie $1(1!) + \dots + k(k!) + (k+1)(k+1)! = (k+2)! - 1$

LHS $= (k+1)! - 1 + (k+1)(k+1)!$
$= (k+1)! \left[1 + k + 1 \right] - 1$
$= (k+1)! \, (k+2) - 1$
$= (k+2)! - 1$
$=$ RHS

$\therefore$ If the statement is true for $n = k$ it is

true for $n=k+1$. Since it is true for $n=1$, it must is true for $n=2$ and so on.
$\therefore$ the statement is true for all positive integers n.

4

b) If $x = 4 - 2\sin^2 t$

$x = 4 - (1 - \cos 2t)$

$x = 3 + \cos 2t \Rightarrow \cos 2t = x - 3$

(i) $\dot{x} = -2\sin 2t$

$\ddot{x} = -4\cos 2t$

$= -4(x-3)$

2

$\therefore$ the motion is of the form

$\ddot{X} = -n^2 X$ where $n=2$, $X = x-3$

Hence the motion is SHM

(ii) Period $= \frac{2\pi}{n}$

$= \frac{2\pi}{2}$

$= \pi$

1

Centre of motion is $x = 3$

1

(iii) If $\dot{x} = -2\sin 2t$

$v^2 = 4\sin^2 2t$

$v^2 = 4(1 - \cos^2 2t)$

$v^2 = 4(1 - (x-3)^2)$

2

$v^2 = 4(1 - x^2 + 6x - 9)$

$v^2 = 4(-8 + 6x - x^2)$

c) (i) $x = vt\cos\alpha$

$y = vt\sin\alpha - \frac{1}{2}gt^2$

When $y = 0$

$vt\sin\alpha - \frac{1}{2}gt^2 = 0$

$$t\left(V\sin\alpha - \tfrac{1}{2}gt\right) = 0$$

$t = 0$ or $V\sin\alpha = \tfrac{1}{2}gt$

$$t = \frac{2V\sin\alpha}{g}$$

When $t = \dfrac{2V\sin\alpha}{g}$

$$x = V \cdot \frac{2V\sin\alpha}{g} \cdot \cos\alpha$$

2

$$= \frac{V^2}{g} \cdot 2\sin\alpha\cos\alpha$$

$$= \frac{V^2}{g} \cdot \sin 2\alpha$$

ii) Let the distance between the cannon & the target be T

$$T + 50 = \frac{V^2 \sin 90^\circ}{g}$$

$$T + 50 = \frac{V^2}{g}$$

Also $T - 20 = \dfrac{V^2 \sin 60^\circ}{g}$

$$T - 20 = \frac{V^2}{g} \cdot \frac{\sqrt{3}}{2}$$

$$T - 20 = \frac{V^2\sqrt{3}}{2g}$$

$$T - 20 = \sqrt{3} \cdot \frac{V^2}{2g}$$

$$T - 20 = \frac{\sqrt{3}}{2}(T + 50)$$

So $\quad 2T - 40 = \sqrt{3}T + 50\sqrt{3}$

$$2T - \sqrt{3}T = 50\sqrt{3} + 40$$

$$T(2-\sqrt{3}) = 50\sqrt{3} + 40$$

$$T = \frac{50\sqrt{3}+40}{2-\sqrt{3}}$$

$$T \doteqdot 472.49\,m \ (2\ d.p)$$

3